Approximation results and subspace correction algorithms for implicit variational inequalities

Lori Badea, Marius Cocou

Approximation results and subspace correction algorithms for implicit variational inequalities

Lori Badea
Institute of Mathematics of the Romanian Academy, Bucharest, Romania

Marius Cocou[1]
Aix-Marseille University, CNRS, LMA UPR 7051, Centrale Marseille, France

Keywords
Evolution problems, implicit variational inequalities, approximation and existence results, subspace correction, domain decomposition, Schwarz method, multilevel methods.

Abstract

This paper deals with the mathematical analysis and the subspace approximation of a system of variational inequalities representing a unified approach to several quasistatic contact problems in elasticity. Using an implicit time discretization scheme and some estimates, convergence properties of the incremental solutions and existence results are presented for a class of abstract implicit evolution variational inequalities involving a nonlinear operator. To solve the corresponding semi-discrete and the fully discrete problems, some general subspace correction algorithms are proposed, for which global convergence is analyzed and error estimates are established.

1 Introduction

This work is concerned with the mathematical analysis and the approximation of a system of evolution variational inequalities representing a unified approach to several quasistatic contact problems in elasticity.

The general results presented here constitute a generalization of the cases studied in [7] and [9] and can be applied to various quasistatic contact problems, including unilateral contact with nonlocal friction, normal compliance

[1]**Corresponding author**:
Marius Cocou, Laboratoire de Mécanique et d'Acoustique C.N.R.S., 31 chemin Joseph Aiguier, 13402 Marseille Cedex 20, France.
Email: cocou@lma.cnrs-mrs.fr

conditions with friction or more complex interaction laws, as, for example, interface laws coupling unilateral contact, adhesion and nonlocal friction between two elastic bodies [17].

Using an implicit time discretization scheme and some estimates of the incremental solutions, approximation and existence results are presented for a class of abstract implicit evolution variational inequalities involving a non-linear operator.

To solve the general elliptic quasi-variational inequalities of the second kind that are obtained by the previous incremental procedure, some subspace correction algorithms are proposed, for which global convergence is analyzed and error estimates are established.

If the subspaces are the finite element spaces of the fine grid associated with a decomposition of the domain, or with the space corresponding to the coarse mesh, these algorithms are in fact one- and two-level Schwarz methods. In this case, we are able to write the convergence rate depending on the overlapping and mesh parameters. Following this way, we can show that our methods have an optimal convergence rate, i.e. their convergence is the same as in the case of linear equations.

Schwarz methods are widely applied for solving linear problems, because they provide robust and efficient solution methods but their generalization to non-linear problems as, for example, quasi-variational inequalities, is not straightforward. In particular, gaining an estimate for the convergence speed of a two-level or multilevel Schwarz method in the case of non-linear problems is far from trivial.

The methods we deal here generalize the iterative method suggested in [15] and [16] for complementarity problems. For these problems, this projected multilevel relaxation was later developed in [12]–[14] and named as monotone multigrid method. On the other hand, the application of this method to other types of convex sets in general abstract spaces and monotone minimizing functionals have been investigated in [1] and [2], for instance. Also, the case where the inequality contains extra terms which do not stem from the minimization of a functional has been investigated in [3]. Additional non-linear terms have also to be considered in the case of quasi-variational, or implicit, inequalities.

Let us finally emphasize that in our opinion, the conditions for a global and optimal convergence rate of the methods with more than two levels (multilevel or multigrid methods) are not yet well understood and that their theoretical understanding will likely require a different approach.

The paper is organized as follows. In Section 2, using an implicit time discretization scheme and some estimates, convergence properties are given and an existence result is established.

In Section 3, general convergence results, based on an semi-discrete internal approximation, are presented.

In Section 4, some subspace correction algorithms are introduced for the solution of the problem discretized in time. These algorithms represent one- and two-level Schwarz methods for the finite element form of the problem. First, three algorithms are introduced in a general framework in a Hilbert space. We introduce here an assumption on the convex set and the correction subspaces, which will be useful in the proof of the convergence of the algorithms. Mainly, this hypothesis refers to the decomposition of the elements in the convex set, and introduces a constant C_0 which will play an important role in the writing of the convergence rate. Another hypothesis is made on the non differentiable term in the inequality. Under these assumptions, we prove that the three subspace algorithms are convergent and give an estimation of the convergence rate. The reminder of the section is devoted to the one- and two-level methods. If we associate the correction subspaces to a domain decomposition, the abstract algorithms are Schwarz methods. We show that the assumptions introduced in the general theory hold and explicitly write the constant C_0 depending on the mesh and domain decomposition parameters. In this way, we get that the convergence rates of the one- and two-level methods for our quasi-variational inequalities are similar with the convergence rates obtained for equations, i.e., we get an optimal convergence. In the case of the two-level methods, the convergence rate is almost independent of the mesh and domain decomposition parameters.

2 A system of implicit evolution inequalities

Let $(V, \langle . , . \rangle)$, $(H, (\cdot, \cdot)_H)$ be two real Hilbert spaces with the associated norms $\| \cdot \|$ and $\| \cdot \|_H$, respectively. Let K be a non empty closed convex cone contained in V with its vertex at 0 and $(K(g))_{g \in V}$ be a family of nonempty convex subsets of K satisfying the following conditions: $0 \in K(0)$ and

$$\text{if } g_n \to g \text{ in } V,\ v_n \in K(g_n) \text{ and } v_n \rightharpoonup v \text{ in } V \text{ then } v \in K(g). \tag{1}$$

We consider a functional $F : V \to \mathbb{R}$ Gateaux differentiable on V and assume that there exist two constants $\alpha,\ \beta > 0$ for which

$$\alpha \|v - u\|^2 \le \langle F'(v) - F'(u), v - u \rangle \tag{2}$$

and

$$\|F'(v) - F'(u)\|_{V'} \le \beta \|v - u\| \tag{3}$$

for all $u, v \in V$, where F' is the Gateaux derivative of F.

Using the relations

$$F(v)-F(u)=\int_0^1 \langle F'(u+r(v-u)),v-u\rangle dr$$
$$=\langle F'(u),v-u\rangle+\int_0^1 \langle F'(u+r(v-u))-F'(u),v-u\rangle dr$$

and (2), (3), it is easily seen that for all $u, v \in V$ we have

$$\begin{aligned}&\langle F'(u),v-u\rangle+\frac{\alpha}{2}\|v-u\|^2 \le F(v)-F(u)\\ \le&\langle F'(u),v-u\rangle+\frac{\beta}{2}\|v-u\|^2.\end{aligned} \tag{4}$$

Since F satisfies (4), it follows that F is strictly convex, sequentially weakly lower semicontinuous and differentiable on V.

We assume that for all $g \in V$ there exists an operator $\gamma(g,\cdot) : K(g) \to H$ such that $\gamma(0,0)=0$,

$$\begin{aligned}&\text{if } \ g_n \to g \ \text{ in } V, \ v_n \in K(g_n) \text{ and } \ v_n \rightharpoonup v \text{ in } V\\ &\text{then } \ \gamma(g_n,v_n) \to \gamma(g,v) \ \text{ in } H\end{aligned} \tag{5}$$

and for all $g_i \in V$, $v_i \in K(g_i)$, $i=1,\,2$,

$$\|\gamma(g_1,v_1)-\gamma(g_2,v_2)\|_H \le k_1(\|g_1-g_2\|+\|v_1-v_2\|), \tag{6}$$

where k_1 is a positive constant.

For all $g \in V$, let $j(g,\cdot,\cdot) : K(g)\times V \to \mathbb{R}$ be a functional satisfying the following conditions:

$$j(g,v,\cdot) \text{ is sequentially weakly continuous on } V \quad \forall\, g \in V,\, v \in K(g), \tag{7}$$

$$j(g,v,\cdot) \ \text{ is sub-additive for all } \ g \in V,\, v \in K(g), \ \text{ that is} \tag{8}$$
$$j(g,v,w_1+w_2) \le j(g,v,w_1)+j(g,v,w_2) \quad \forall\, g,\, w_{1,2} \in V,\, v \in K(g),$$

$$j(g,v,\cdot) \ \text{ is positively homogeneous for all } \ g \in V,\, v \in K(g), \tag{9}$$
$$\text{that is } \ j(g,v,\theta w)=\theta j(g,v,w) \quad \forall\, g,\, w \in V,\, v \in K(g),\, \theta \ge 0,$$

$$j(0,0,w)=0 \quad \forall\, w \in V, \tag{10}$$

and there exists $k_2 > 0$ such that

$$\begin{aligned} & |j(g_1, v_1, w_2) + j(g_2, v_2, w_1) - j(g_1, v_1, w_1) - j(g_2, v_2, w_2)| \\ \leq & k_2(\|g_1 - g_2\| + \|\gamma(g_1, v_1) - \gamma(g_2, v_2)\|_H)\|w_1 - w_2\| \\ & \forall\, g_i,\, w_i \in V,\, v_i \in K(g_i),\, i = 1,\, 2. \end{aligned} \tag{11}$$

We assume that k_1 and k_2 satisfy the following property:

$$k_1 k_2 < \alpha. \tag{12}$$

For all $g \in V$, we consider a functional $b(g, \cdot, \cdot) : K(g) \times V \to \mathbb{R}$ such that

$$\forall\, g \in V,\, v \in K(g),\ b(g, v, \cdot) \text{ is linear and continuous on } V \tag{13}$$

and

$$\begin{aligned} & |b(g_1, v_1, w) - b(g_2, v_2, w)| \leq k_b(\|g_1 - g_2\| \\ & + \|v_1 - v_2\|)\|w\| \quad \forall\, g_i,\, w \in V,\, v_i \in K(g_i),\, i = 1,\, 2, \end{aligned} \tag{14}$$

where k_b is a positive constant. From the above properties of F, j and K and by a classical argument, it follows that for all $g \in V$, $d \in K$, $w \in K(g)$ the elliptic variational inequality

$$u \in K \quad \langle F'(u), v - u\rangle + j(g, w, v - d) - j(g, w, u - d) \geq 0 \quad \forall\, v \in K$$

has a unique solution, so that we can define the mapping $S_{g,d} : K(g) \to K$ by $S_{g,d}(w) = u$. We assume that for all $g \in V$, $d \in K$

$$K(g) \text{ is stable under } S_{g,d} \text{ i.e. } S_{g,d}(K(g)) \subset K(g). \tag{15}$$

For all $g \in V$, $d \in K$, we consider the following problems:

$$(\tilde{P}) \begin{cases} u \in K(g) \quad \langle F'(u), v - u\rangle + j(g, u, v - d) - j(g, u, u - d) \\ \geq b(g, u, v - u) \quad \forall\, v \in V, \\ b(g, u, z - u) \geq 0 \quad \forall\, z \in K, \end{cases}$$

$$(\tilde{Q})\ u \in K(g) \quad \langle F'(u), v - u\rangle + j(g, u, v - d) - j(g, u, u - d) \geq 0 \quad \forall\, v \in K,$$

and we assume that

$$\text{if } u \text{ is a solution of } (\tilde{Q}), \text{ then } u \text{ is a solution of } (\tilde{P}). \tag{16}$$

Remark 1. If u satisfies $(\tilde{P})$, then u obviously satisfies $(\tilde{Q})$.

Let $f \in W^{1,2}(0,T;V)$ be given. Using the hypotheses (2), (3), (15), (11) and (6), it follows that $S_{f(0),0} : K(f(0)) \to K(f(0))$ is a contraction if the condition (12) holds. Thus, the following implicit elliptic variational inequality has a unique solution $u_0 \in K(f(0))$ (see, e.g. [6]):

$$\langle F'(u_0), w - u_0\rangle + j(f(0), u_0, w) - j(f(0), u_0, u_0) \geq 0 \quad \forall\, w \in K. \tag{17}$$

We consider the following evolution problems, involving implicit variational inequalities.

Problem P: Find $u \in W^{1,2}(0,T;V)$ such that

$$(P) \quad \begin{cases} u(0) = u_0,\ u(t) \in K(f(t)) \quad \forall\, t \in]0,T[, \\ \langle F'(u(t)), v - \dot{u}(t)\rangle + j(f(t), u(t), v) - j(f(t), u(t), \dot{u}(t)) \\ \ \geq b(f(t), u(t), v - \dot{u}(t)) \quad \forall\, v \in V \text{ a.e. on }]0,T[, \\ b(f(t), u(t), w - u(t)) \geq 0 \quad \forall\, w \in K,\ \forall\, t \in]0,T[, \end{cases}$$

Problem Q: Find $u \in W^{1,2}(0,T;V)$ such that

$$(Q) \quad \begin{cases} u(0) = u_0,\ u(t) \in K(f(t)) \quad \forall\, t \in]0,T[, \\ \langle F'(u(t)), w - u(t)\rangle + j(f(t), u(t), w - u(t) + \dot{u}(t)) \\ \ - j(f(t), u(t), \dot{u}(t)) \geq 0 \quad \forall\, w \in K \text{ a.e. on }]0,T[, \end{cases}$$

and

Problem $\hat{\mathbf{Q}}$: Find $u \in W^{1,2}(0,T;V)$ such that

$$(\hat{Q}) \quad \begin{cases} u(0) = u_0,\ u(t) \in K(f(t)) \quad \forall\, t \in]0,T[, \\ F(w) - F(u(t)) + j(f(t), u(t), w - u(t) + \dot{u}(t)) \\ \ - j(f(t), u(t), \dot{u}(t)) \geq \dfrac{\alpha}{2}\|w - u(t)\|^2 \quad \forall\, w \in K \text{ a.e. on }]0,T[. \end{cases}$$

Remark 2. i) By (16) and Remark 1 with $d = u - \dot{u}$, the problems P and Q are equivalent.
ii) Since F satisfies (4) and using the definition of the Gateaux derivative of F with the convexity of $j(f, u, \cdot)$, it follows that the problems Q and $\hat{Q}$ are equivalent.

We shall prove the existence of a solution to problem P by using an implicit time discretization scheme and its convergence properties.

For $\nu \in N^*$, we set $\Delta t := T/\nu$, $t_\iota := \iota\,\Delta t$ and $K^\iota := K(f(t_\iota))$, $\iota = 0, 1, ..., \nu$. If θ is a continuous function of $t \in [0,T]$ valued in some vector space, we use the notations $\theta^\iota := \theta(t_\iota)$ unless $\theta = u$, and if ζ^ι, $\forall\, \iota \in \{0, 1, ..., \nu\}$, are elements of some vector space, then we set

$$\partial\zeta^\iota := \frac{\zeta^{\iota+1} - \zeta^\iota}{\Delta t} \quad \forall\, \iota \in \{0, 1, ..., \nu - 1\}.$$

We denote $u^0 := u_0$ and we approximate (P) using the following sequence of incremental problems $(P_\nu^\iota)_{\iota=0,1,...,\nu-1}$.
Problem $\mathbf{P}_\nu^\iota$: Find $u^{\iota+1} \in K^{\iota+1}$ such that

$$(P_\nu^\iota) \quad \begin{cases} \langle F'(u^{\iota+1}), v - \partial u^\iota \rangle + j(f^{\iota+1}, u^{\iota+1}, v) - j(f^{\iota+1}, u^{\iota+1}, \partial u^\iota) \\ \geq b(f^{\iota+1}, u^{\iota+1}, v - \partial u^\iota) \quad \forall\, v \in V, \\ b(f^{\iota+1}, u^{\iota+1}, w - u^{\iota+1}) \geq 0 \quad \forall\, w \in K. \end{cases}$$

By (16) and Remark 1 for $g = f^{\iota+1}$, $d = u^\iota$, and using similar arguments as in Remark 2 ii), respectively, it is easily seen that for all $\iota \in \{0, 1, ..., \nu - 1\}$ the problem P_ν^ι is equivalent to each of the following variational inequalities: find $u^{\iota+1} \in K^{\iota+1}$ such that

$$(Q_\nu^\iota) \quad \begin{cases} \langle F'(u^{\iota+1}), w - u^{\iota+1} \rangle + j(f^{\iota+1}, u^{\iota+1}, w - u^\iota) \\ - j(f^{\iota+1}, u^{\iota+1}, u^{\iota+1} - u^\iota) \geq 0 \quad \forall\, w \in K, \end{cases}$$

find $u^{\iota+1} \in K^{\iota+1}$ such that

$$(\hat{Q}_\nu^\iota) \quad \begin{cases} F(w) - F(u^{\iota+1}) + j(f^{\iota+1}, u^{\iota+1}, w - u^\iota) \\ - j(f^{\iota+1}, u^{\iota+1}, u^{\iota+1} - u^\iota) \geq \dfrac{\alpha}{2}\|w - u^{\iota+1}\|^2 \quad \forall\, w \in K. \end{cases}$$

From the hypotheses (2), (12), (15), (11) and (6), it follows that $S_{f^{\iota+1},u^\iota} : K^{\iota+1} \to K^{\iota+1}$ is a contraction. Therefore (Q_ν^ι) has a unique solution which is equally the unique solution of (P_ν^ι) and of $(\hat{Q}_\nu^\iota)$, for all $\iota \in \{0, 1, ..., \nu - 1\}$.

Let us define the following functions:

$$\begin{cases} u_\nu(0) = \hat{u}_\nu(0) = u^0, \ f_\nu(0) = f^0 \ \text{ and} \\ \forall\, \iota \in \{0, 1, ..., \nu - 1\}, \ \forall\, t \in]t_\iota, t_{\iota+1}], \\ u_\nu(t) = u^{\iota+1}, \\ \hat{u}_\nu(t) = u^\iota + (t - t_\iota)\partial u^\iota, \\ f_\nu(t) = f^{\iota+1}. \end{cases}$$

Then for all $\nu \in N^*$ the sequence of inequalities $(P_\nu^\iota)_{\iota=0,1,...,\nu-1}$ is equivalent to the following incremental formulation: for almost every $t \in [0, T]$

$$(P_\nu) \begin{cases} u_\nu(t) \in K(f_\nu(t)), \ \langle F'(u_\nu(t)), v - \dfrac{d}{dt}\hat{u}_\nu(t) \rangle + j(f_\nu(t), u_\nu(t), v) \\ - j(f_\nu(t), u_\nu(t), \dfrac{d}{dt}\hat{u}_\nu(t)) \geq b(f_\nu(t), u_\nu(t), v - \dfrac{d}{dt}\hat{u}_\nu(t)) \ \ \forall\, v \in V, \\ b(f_\nu(t), u_\nu(t), w - u_\nu(t)) \geq 0 \quad \forall\, w \in K. \end{cases}$$

The existence of a solution of the problem P will be proved by using the following results and their proofs will be presented in a forthcoming paper.

Lemma 2.1. *There exist a subsequence of* $(u_\nu, \hat{u}_\nu)_\nu$*, denoted by* $(u_{\nu_p}, \hat{u}_{\nu_p})_p$*, and an element* $u \in W^{1,2}(0,T;V)$ *such that*

$$u_{\nu_p}(t) \rightharpoonup u(t) \quad in \quad V \quad \forall\, t \in [0,T], \tag{18}$$

$$\hat{u}_{\nu_p} \rightharpoonup u \quad in \quad W^{1,2}(0,T;V), \tag{19}$$

$$\frac{d}{dt}\hat{u}_\nu \rightharpoonup \dot{u} \quad in \quad L^2(0,T;V). \tag{20}$$

Also, for all $s \in [\,0,T\,]$*, we have* $u(s) \in K(f(s))$ *and*

$$\liminf_{p\to\infty} \int_0^s j(f_{\nu_p}(t), u_{\nu_p}(t), \frac{d}{dt}\hat{u}_{\nu_p}(t))\,dt \geq \int_0^s j(f(t), u(t), \dot{u}(t))\,dt. \tag{21}$$

We have the following strong convergence and existence result.

Theorem 2.2. *Under the assumptions (1)–(3), (5)–(16) every convergent subsequence of* $(u_\nu, \hat{u}_\nu)_\nu$*, still denoted by* $(u_\nu, \hat{u}_\nu)_\nu$*, and its limit* $u \in W^{1,2}(0,T;V)$*, given by lemma 2.1, satisfy the following properties:*

$$u_\nu(t) \to u(t) \quad in \quad V \quad \forall\, t \in [0,T], \tag{22}$$

$$\hat{u}_\nu \to u \quad in \quad L^2(0,T;V), \tag{23}$$

and u *is a solution of problem* P*.*

3 Internal approximation and convergence results

In this section we shall study the approximation of problem P by using a convergence result for a method based on an internal approximation and a backward difference scheme. The full proofs will be presented in a forthcoming paper.

First, we consider a semi-discrete approximation of (P), which extends the classical internal approximations as presented in, e.g. [11], [10]. For a positive parameter h converging to 0, let $(V_h)_h$ be an internal approximation of V, that is a family of finite-dimensional subspaces of V which satisfies:

$$\begin{aligned} &\text{there exist } U \subset V \text{ such that } \overline{U} = V \text{ and} \\ &\forall\, v \in U,\, \exists\, v_h \in V_h \ \text{ for each } h, \ \text{ such that } \ v_h \to v \ \text{ in } V. \end{aligned} \tag{24}$$

Let $(K_h)_h$ be a family of convex cones with their vertices at 0 such that $K_h \subset V_h$ for all h and $(K_h)_h$ is an internal approximation of K, i.e.

$$\text{if } \; v_h \in K_h \; \text{ for all } h \; \text{ and } v_h \rightharpoonup v \text{ then } v \in K, \tag{25}$$

$$\forall\, v \in K,\, \exists\, v_h \in K_h \; \text{ for each } h, \text{ such that } \; v_h \to v \; \text{ in } V. \tag{26}$$

Let $(K_h(g))_{g\in V}$ be a family of nonempty convex subsets of K_h such that $0 \in K_h(0)$ for all h, satisfying the following conditions:

$$\text{if } \; g_n \to g \; \text{ in } V,\, v_{hn} \in K_h(g_n) \; \text{ and } \; v_{hn} \to v_h \text{ in } V_h \; \text{ then } \; v_h \in K_h(g), \tag{27}$$

$$\text{if } \; v_h \in K_h(g) \; \text{ for all } h \; \text{ and } v_h \rightharpoonup v \text{ then } v \in K(g) \;\; \forall\, g \in V. \tag{28}$$

We assume that for all $g \in V$ there exists an operator $\gamma_h(g,\cdot) : K_h(g) \to H$ such that $\gamma_h(0,0) = 0$ and for all $g_i \in V$, $v_{hi} \in K_h(g_i)$, $i = 1, 2$,

$$\|\gamma_h(g_1, v_{h1}) - \gamma_h(g_2, v_{h2})\|_H \le k_1(\|g_1 - g_2\| + \|v_{h1} - v_{h2}\|). \tag{29}$$

For all $g \in V$, let $j_h(g,\cdot,\cdot) : K_h(g) \times V_h \to \mathbb{R}$ be a functional satisfying the following conditions for all $g \in V$:

$$\begin{aligned} &\text{if } v_h \in K_h(g) \text{ for all } h,\; v_h \rightharpoonup v \;\text{ in } V \text{ and } \; w_h \rightharpoonup w \;\text{ in } \; V \\ &\text{then } \lim_{h\to 0} j_h(g, v_h, w_h) = j(g, v, w), \end{aligned} \tag{30}$$

$$\text{for all } h \text{ and } v_h \in K_h(g) \quad j_h(g, v_h, \cdot) \; \text{ is sub-additive,} \tag{31}$$

$$\text{for all } h \text{ and } v_h \in K_h(g) \quad j_h(g, v_h, \cdot) \; \text{ is positively homogeneous,} \tag{32}$$

$$j_h(0, 0, w_h) = 0 \quad \forall\, w_h \in V_h, \tag{33}$$

and

$$\begin{aligned} &\text{if } v_h(t) \in K_h(g(t)) \; \text{ for all } h \text{ and } t \in [0,T],\; v_h \rightharpoonup v \; \text{ in } W^{1,2}(0,T;V) \\ &\text{then } \liminf_{h\to 0} \int_0^T j_h(g(t), v_h(t), \dot v_h(t))\, dt \ge \int_0^T j(g(t), v(t), \dot v(t))\, dt \\ &\qquad \text{for all } g \in C([0,T];V), \end{aligned} \tag{34}$$

$$\begin{aligned} &|j_h(g_1, v_{h1}, w_{h2}) + j_h(g_2, v_{h2}, w_{h1}) - j_h(g_1, v_{h1}, w_{h1}) - j_h(g_2, v_{h2}, w_{h2})| \\ &\quad \le \; k_2(\|g_1 - g_2\| + \|\gamma_h(g_1, v_{h1}) - \gamma_h(g_2, v_{h2})\|_H)\|w_{h1} - w_{h2}\| \\ &\qquad \forall\, g_i \in V,\; v_{hi} \in K_h(g_i),\; w_{hi} \in V_h,\; i = 1, 2. \end{aligned} \tag{35}$$

From the properties of F, j_h and K_h, it follows that for all $g \in V$, $d_h \in K_h$, $w_h \in K_h(g)$, the elliptic variational inequality: $u_h \in K_h$

$$\langle F'(u_h), v_h - u_h \rangle + j_h(g, w_h, v_h - d_h) - j_h(g, w_h, u_h - d_h) \geq 0 \quad \forall\, v_h \in K_h$$

has a unique solution. Hence we can define a mapping $S^h_{g,d_h} : K_h(g) \to K_h$ by $S^h_{g,d_h}(w_h) = u_h$. We suppose that for all $g \in V$, $d_h \in K_h$

$$S^h_{g,d_h}(K_h(g)) \subset K_h(g). \tag{36}$$

For all $g \in V$, $d_h \in K_h$, we consider the following problems:

$$(\tilde{P}_h) \quad \begin{cases} u_h \in K_h(g), \;\; \langle F'(u_h), v_h - u_h \rangle + j_h(g, u_h, v_h - d_h) \\ - j_h(g, u_h, u_h - d_h) \geq b(g, u_h, v_h - u_h) \quad \forall\, v_h \in V_h, \\ b(g, u_h, z_h - u_h) \geq 0 \quad \forall\, z_h \in K_h, \end{cases}$$

and

$$(\tilde{Q}_h) \quad \begin{cases} u_h \in K_h(g), \;\; \langle F'(u_h), v_h - u_h \rangle + j_h(g, u_h, v_h - d_h) \\ - j_h(g, u_h, u_h - d_h) \geq 0 \quad \forall\, v_h \in K_h. \end{cases}$$

We assume that

$$\text{if } u_h \text{ is a solution of } (\tilde{Q}_h), \text{ then } u_h \text{ is a solution of } (\tilde{P}_h). \tag{37}$$

Remark 3. If u_h satisfies $(\tilde{P}_h)$, then u_h obviously satisfies $(\tilde{Q}_h)$.

Now, we consider the following equivalent semi-discrete problems.
Problem $\mathbf{P_h}$: Find $u_h \in W^{1,2}(0,T;V_h)$ such that

$$(P_h) \quad \begin{cases} u_h(0) = u_{0h}, \;\; u_h(t) \in K_h(f(t)) \quad \forall\, t \in]0,T[, \\ \langle F'(u_h(t)), v_h - \dot{u}_h(t) \rangle + j_h(f(t), u_h(t), v_h) - j_h(f(t), u_h(t), \dot{u}_h(t)) \\ \geq b(f(t), u_h(t), v_h - \dot{u}_h(t)) \quad \forall\, v_h \in V_h \text{ a.e. on }]0,T[, \\ b(f(t), u_h(t), z_h - u_h(t)) \geq 0 \quad \forall\, z_h \in K_h, \quad \forall\, t \in]0,T[, \end{cases}$$

Problem $\mathbf{Q_h}$: Find $u_h \in W^{1,2}(0,T;V_h)$ such that

$$(Q_h) \quad \begin{cases} u(0) = u_{0h}, \; u_h(t) \in K_h(f(t)) \quad \forall\, t \in]0,T[, \\ \langle F'(u_h(t)), w_h - u_h(t) \rangle + j_h(f(t), u_h(t), w_h - u_h(t) + \dot{u}_h(t)) \\ - j_h(f(t), u_h(t), \dot{u}_h(t)) \geq 0 \quad \forall\, w_h \in K_h \text{ a.e. on }]0,T[, \end{cases}$$

and

Problem $\hat{\mathbf{Q}}_{\mathbf{h}}$: Find $u_h \in W^{1,2}(0,T;V_h)$ such that

$$(\hat{Q}_h)\ \begin{cases} u(0) = u_{0h},\, u_h(t) \in K_h(f(t)) \quad \forall\, t \in]0,T[, \\ F(w_h) - F(u_h(t)) + j_h(f(t), u_h(t), w_h - u_h(t) + \dot{u}_h(t)) \\ -\, j_h(f(t), u_h(t), \dot{u}_h(t)) \geq \dfrac{\alpha}{2}\|w_h - u_h(t)\|^2 \quad \forall\, w_h \in K_h \text{ a.e. on }]0,T[. \end{cases}$$

where $u_{0h} \in K_h(f(0))$ is the unique solution of the variational inequality

$$\begin{aligned} &\langle F'(u_{0h}), w_h - u_{0h}\rangle + j_h(f(0), u_{0h}, w_h) \\ &-\, j_h(f(0), u_{0h}, u_{0h}) \geq 0 \quad \forall\, w_h \in K_h. \end{aligned} \tag{38}$$

The full discretization of (P_h) is obtained by using an implicit scheme as in Section 2 for (P). For $u_h^0 := u_{0h}$ and $\iota \in \{0,1,...,\nu-1\}$, we define $u_h^{\iota+1}$ as the solution of the following problem.
Problem $\mathbf{P}^{\iota}_{\mathbf{h}\nu}$: Find $u_h^{\iota+1} \in K_h^{\iota+1}$ such that

$$(P^{\iota}_{h\nu})\ \begin{cases} \langle F'(u_h^{\iota+1}), v_h - \partial u_h^{\iota}\rangle + j_h(f^{\iota+1}, u_h^{\iota+1}, v_h) - j_h(f^{\iota+1}, u_h^{\iota+1}, \partial u_h^{\iota}) \\ \geq b(f^{\iota+1}, u_h^{\iota+1}, v_h - \partial u_h^{\iota}) \quad \forall\, v_h \in V_h, \\ b(f^{\iota+1}, u_h^{\iota+1}, z_h - u_h^{\iota+1}) \geq 0 \quad \forall\, z_h \in K_h, \end{cases}$$

where $K_h^{\iota+1} := K_h(f^{\iota+1})$.

As in Section 2, it follows that for all $\iota \in \{0,1,...,\nu-1\}$ the problem $(P^{\iota}_{h\nu})$ is equivalent to each of the following variational inequalities:

find $u_h^{\iota+1} \in K_h^{\iota+1}$ such that

$$(Q^{\iota}_{h\nu})\ \begin{cases} \langle F'(u_h^{\iota+1}), w_h - u_h^{\iota+1}\rangle + j_h(f^{\iota+1}, u_h^{\iota+1}, w_h - u_h^{\iota}) \\ -\, j_h(f^{\iota+1}, u_h^{\iota+1}, u_h^{\iota+1} - u_h^{\iota}) \geq 0 \quad \forall\, w_h \in K_h, \end{cases}$$

find $u_h^{\iota+1} \in K_h^{\iota+1}$ such that

$$(\hat{Q}^{\iota}_{h\nu})\ \begin{cases} F(w_h) - F(u_h^{\iota+1}) + j_h(f^{\iota+1}, u_h^{\iota+1}, w_h - u_h^{\iota}) \\ -\, j_h(f^{\iota+1}, u_h^{\iota+1}, u_h^{\iota+1} - u_h^{\iota}) \geq \dfrac{\alpha}{2}\|w_h - u_h^{\iota+1}\|^2 \quad \forall\, w_h \in K_h. \end{cases}$$

From the relations (2), (29), (35), (12) and (36) it follows that the mapping $S^h_{f^{\iota+1}, u_h^{\iota}} : K_h^{\iota+1} \to K_h^{\iota+1}$ is a contraction, so that $(Q^{\iota}_{h\nu})$ has a unique solution which is also the unique solution of $(P^{\iota}_{h\nu})$ and of $(\hat{Q}^{\iota}_{h\nu})$, for all $\iota \in \{0,1,...,\nu-1\}$.

If we define the functions

$$\begin{cases} u_{h\nu}(0) = \hat{u}_{h\nu}(0) = u_{0h} \quad \text{and} \\ \forall \iota \in \{0, 1, ..., \nu - 1\}, \quad \forall t \in]t_\iota, t_{\iota+1}], \\ \ u_{h\nu}(t) = u_h^{\iota+1}, \\ \ \hat{u}_{h\nu}(t) = u_h^\iota + (t - t_\iota)\partial u_h^\iota, \end{cases}$$

then for all $\nu \in N^*$ the sequence of inequalities $(P_\nu^{h\iota})_{\iota=0,1,...,\nu-1}$ is equivalent to the following incremental formulation: for almost every $t \in [0, T]$

$$(P_{h\nu}) \begin{cases} u_{h\nu}(t) \in K_h(f_\nu(t)), \ \langle F'(u_{h\nu}(t)), v_h - \dfrac{d}{dt}\hat{u}_{h\nu}(t)\rangle + j_h(f_\nu(t), u_{h\nu}(t), v_h) \\ -\, j_h(f_\nu(t), u_{h\nu}(t), \dfrac{d}{dt}\hat{u}_{h\nu}(t)) \geq b(f_\nu(t), u_{h\nu}(t), v_h - \dfrac{d}{dt}\hat{u}_{h\nu}(t)) \ \forall v_h \in V_h, \\ b(f_\nu(t), u_{h\nu}(t), w_h - u_{h\nu}(t)) \geq 0 \quad \forall w_h \in K_h. \end{cases}$$

We have the analogue to theorem 2.2 in the finite-dimensional case.

Theorem 3.1. *Assume that (2), (3), (13), (14), (27), (29), (31)–(33), (35)–(37) hold. Then there exists a subsequence of* $(u_{h\nu}, \hat{u}_{h\nu})_\nu$*, still denoted by* $(u_{h\nu}, \hat{u}_{h\nu})_\nu$*, such that*

$$u_{h\nu}(t) \to u_h(t) \quad in \quad V \quad \forall t \in [0, T], \tag{39}$$

$$\hat{u}_{h\nu} \to u_h \quad in \quad L^2(0, T; V), \tag{40}$$

where u_h *is a solution of* (P_h)*.*

Using the previous theorem and a priori estimates for the solutions of (P_h) we have the following convergence and existence result.

Theorem 3.2. *Under the assumptions (2), (3), (13), (14), (24)–(37) there exists a subsequence of* $(u_h)_h$ *such that*

$$u_h(t) \to u(t) \quad in \quad V \quad \forall t \in [0, T], \tag{41}$$

$$u_h \to u \quad in \quad L^2(0, T; V), \tag{42}$$

$$\dot{u}_h \rightharpoonup \dot{u} \quad in \quad L^2(0, T; V), \tag{43}$$

where u *is a solution of* (P)*.*

Using theorems 3.1 and 3.2, we obtain the following main approximation result.

Theorem 3.3. *Under the assumptions of theorem 3.2, there exists a subsequence of* $(u_{h\nu})_{h\nu}$ *such that*

$$u_{h\nu}(t) \to u(t) \quad in \quad V \quad \forall\, t \in [0,T], \tag{44}$$

$$\dot{u}_{h\nu} \rightharpoonup \dot{u} \quad in \quad L^2(0,T;V), \tag{45}$$

where $u \in W^{1,2}(0,T;V)$ *is a solution of* (P).

Furthermore any cluster point of $(u_{h\nu})_{h\nu}$ *is a solution of* (P).

4 Subspace correction approximation

The aim of this section is to give, for a fixed time step ι, one- and two-level Schwarz methods for problem (Q_ν^ι), prove their global convergence and estimate the convergence rate. First, we present three correction algorithms in an abstract Hilbert space, prove, under some assumptions, their global convergence and estimate the error. The domain decomposition methods are obtained from these general algorithms by associating to a domain decomposition some subspaces of a Sobolev space. In particular, the one- and two-level methods are obtained using the finite element spaces. In this case, we can estimate the convergence rate depending on the mesh and overlapping parameters.

4.1 Subspace correction algorithms

As in the previous section, we consider a Hilbert space V, and let $V_1, \cdots, V_m$ be some closed subspaces. We also consider a convex subset $\mathcal{K} \subset V$, and assume that it satisfies the following assumption.

Assumption 4.1. There exists a constant C_0 such that for any $w, v \in \mathcal{K}$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in \mathcal{K}$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying

$$w + \sum_{j=1}^{i-1} w_j + v_i \in \mathcal{K} \text{ for } i = 1, \cdots, m, \tag{46}$$

$$v - w = \sum_{i=1}^{m} v_i, \tag{47}$$

and

$$\sum_{i=1}^{m} \|v_i\| \leq C_0 \left(\|v - w\| + \sum_{i=1}^{m} \|w_i\| \right). \tag{48}$$

This assumption looks complicated enough, but we will see in what follows that it is satisfied for many different convex sets in Sobolev spaces, in particular, for the convex sets K and $K(g)$ in the previous section. In the proofs, v is the exact solution, w is the current iterate, and w_i are the corrections on the subspaces V_i, $i = 1, \ldots, m$. In the case where the convex set is written as a sum of convex subsets or the problem has no constraint, (46) and (47) are always satisfied with $w_i = 0$. In these cases, the above assumption is much more simple, and for this reason (48) usually is known without the extra terms given by w_i.

Now, we consider a functional $\varphi : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ and we assume that is convex and lower semicontinuous in the second variable, and

$$\begin{aligned} &|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \\ \leq &k_1 k_2 ||v_1 - v_2|| ||w_1 - w_2||, \ \forall v_1, v_2, w_1, w_2 \in \mathcal{K}. \end{aligned} \tag{49}$$

Also, we suppose that

Assumption 4.2.

$$\begin{aligned} &\sum_{i=1}^{m} [\varphi(u, w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(u, w + \sum_{j=1}^{i-1} w_j + w_i)] \\ \leq &\varphi(u, v) - \varphi(u, w + \sum_{i=1}^{m} w_i) \end{aligned} \tag{50}$$

for any $u \in \mathcal{K}$, and for $v, w \in \mathcal{K}$ and $v_i, w_i \in V_i$, $i = 1, \ldots, m$, in Assumption 4.1.

This assumption has been introduced for proof reasons.

As in the previous section, we take a Gateaux differentiable functional $F : V \to \mathbb{R}$ defined satisfying (2) and (3), we consider the problem of finding $u \in \mathcal{K}$, the solution of the following quasi-variational inequality:

$$\langle F'(u), v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq 0 \quad \forall v \in \mathcal{K}. \tag{51}$$

Since φ satisfies (49), with similar arguments as for problem (Q_ν^ι), we can prove that problem (51) has a solution and it is unique.

Evidently, since φ is convex in the second variable, and F is differentiable and satisfies (4), problem (51) is equivalent with the minimization problem

$$u \in \mathcal{K} : \ F(u) + \varphi(u, u) \leq F(v) + \varphi(u, v) \ \forall v \in \mathcal{K}. \tag{52}$$

Also, in view of (4) we see that the solution u of (51) satisfies

$$\frac{\alpha}{2} \|v - u\|^2 \leq F(v) - F(u) + \varphi(u, v) - \varphi(u, u) \ \forall v \in \mathcal{K}. \tag{53}$$

A first algorithm corresponding to the subspaces $V_1, \cdots, V_m$ and the convex set $\mathcal{K}$ is written as

Algorithm 4.1. We start the algorithm with an arbitrary $u^0 \in \mathcal{K}$. At iteration $n+1$, having $u^n \in \mathcal{K}$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in \mathcal{K}$ satisfying

$$\begin{aligned}
&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle \\
&+ \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + v_i) \\
&- \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \\
&\forall\, v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in \mathcal{K},
\end{aligned} \tag{54}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

A simplified variant of Algorithm 4.1 can be written as

Algorithm 4.2. We start the algorithm with an arbitrary $u^0 \in \mathcal{K}$. At iteration $n+1$, having $u^n \in \mathcal{K}$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in \mathcal{K}$ satisfying

$$\begin{aligned}
&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + v_i) \\
&- \varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \quad \forall\, v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in \mathcal{K},
\end{aligned} \tag{55}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

We can simplify Algorithm 4.1 even more as

Algorithm 4.3. We start the algorithm with an arbitrary $u^0 \in \mathcal{K}$. At iteration $n+1$, having $u^n \in \mathcal{K}$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, the local corrections $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in \mathcal{K}$ satisfying

$$\begin{aligned}
&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^n, u^{n+\frac{i-1}{m}} + v_i) \\
&- \varphi(u^n, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0,\ \forall\, v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in \mathcal{K},
\end{aligned} \tag{56}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

As for problem (51), we can prove that problems (54)–(56) are equivalent with some minimization problems and they have unique solutions.

The following theorem proves that if $k_1 k_2$ is small enough in comparison with α and β, then Algorithms 4.1, 4.2 and 4.3 are convergent.

Theorem 4.1. *Let us assume that Assumptions 4.1 and 4.2 are satisfied. Then, if u is the solution of problem (51), $u^{n+\frac{i}{m}}$, $n \geq 0$, $i = 1, \ldots, m$, are its approximations obtained from one of Algorithms 4.1, 4.2 or 4.3, and*

$$\frac{\alpha}{2} \geq mk_1k_2 + \sqrt{2m(25C_0+8)\beta k_1k_2}, \tag{57}$$

then we have the following error estimations

$$\begin{aligned} &F(u^n) + \varphi(u, u^n) - F(u) - \varphi(u, u) \\ &\leq \left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)\right], \end{aligned} \tag{58}$$

$$\|u^n - u\|^2 \leq \frac{2}{\alpha}\left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)\right], \tag{59}$$

where the constant $C_1 > 0$ depends on α, β, k_1, k_2, the number of subspaces m, and on the constant C_0 introduced in Assumption 4.1.

In a slightly more general form, this theorem has been proved in [4] (see also [5]). In the case of Algorithm 4.1, the constant C_1 can be written as,

$$\begin{aligned} C_1 &= C_2/C_3 \\ C_2 &= \beta m(1 + 2C_0 + \frac{C_0}{\varepsilon_1}) + k_1k_2m(1 + 2C_0 + \frac{1+3C_0}{\varepsilon_2}) \\ C_3 &= \frac{\alpha}{2} - k_1k_2(1+\varepsilon_3)m \end{aligned} \tag{60}$$

where

$$\varepsilon_1 = \varepsilon_2 = \frac{2k_1k_2m}{\frac{\alpha}{2} - k_1k_2m}, \quad \varepsilon_3 = \frac{\frac{\alpha}{2} - k_1k_2m}{2k_1k_2m},$$

For Algorithms 4.2 and 4.3, the constant C_1 has similar expressions.

4.2 Convergence rates for the one- and two-level methods

Algorithms 4.1–4.3 can be viewed as multiplicative Schwarz methods in a subspace correction variant if the solution space is a Sobolev space and associate the subspaces to the subsets in a domain decomposition. Let us assume that the domain Ω is decomposed as

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \tag{61}$$

If for instance $V = H_0^1(\Omega)$ and we take $V_i = H_0^1(\Omega_i)$, $i = 1, \ldots, m$, then the above algorithms are Schwarz methods.

The convergence rates given in Theorem 4.1 depend on the functionals F and φ, the number m of the subspaces and the constant C_0 introduced in Assumption 4.1. The number of subspaces can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends in general on the dimension of the Euclidean space where the domain lies, we can conclude that our convergence rate essentially depends on the constant C_0.

We make now some remarks on the application of Algorithms 4.1–4.3 for the solution of problem (Q_ν^ι).

First, we see that we can take $K^{\iota+1}$ in the place of K in problem (Q_ν^ι), i.e. this problem is equivalent with: find $u^{\iota+1} \in K^{\iota+1}$ such that

$$(R_\nu^\iota) \quad \begin{cases} \langle F'(u^{\iota+1}), w - u^{\iota+1} \rangle + j(f^{\iota+1}, u^{\iota+1}, w - u^\iota) \\ - j(f^{\iota+1}, u^{\iota+1}, u^{\iota+1} - u^\iota) \geq 0 \quad \forall\, w \in K^{\iota+1}. \end{cases}$$

Indeed, since $K^{\iota+1} \subset K$, it follows that the solution of (Q_ν^ι) is also the solution of (R_ν^ι). But, with similar arguments as for (Q_ν^ι), we can prove that problem (R_ν^ι) has a unique solution.

It is proved in [1] that if the convex set $\mathcal{K}$ has the property

Property 4.1. If $v, w \in \mathcal{K}$, and if $\theta \in C^0(\overline{\Omega})$, $\theta \in C^1(\Omega_i)$, $i = 1, \ldots, m$, with $0 \leq \theta \leq 1$, then $\theta v + (1 - \theta) w \in \mathcal{K}$

then Assumption 4.1 is satisfied with a C_0 depending on $1/\delta$, where δ is the overlapping parameter of the domain decomposition (61). In general, the convex sets $\mathcal{K}$ which are defined only by the values of the functions, but not by their derivatives, have the above property. The convex sets of one-obstacle type, like in contact problems (see [7], for instance), or those of two-obstacle type have Property 4.1. Our problems (Q_ν^ι), or (R_ν^ι), are defined in an abstract Hilert space V, and we shall suppose in the following that their convex set $K^{\iota+1}$ has this property.

Since $f^{\iota+1}$ and u^ι are fixed in problem (R_ν^ι), writing

$$\psi(u, v) = j(f^{\iota+1}, u, v - u^\iota) \tag{62}$$

this functional has the properties asked of φ in problem (51), i.e. it is lower semicontinuous and convex in the second variable, and satisfies (49). But, such a ψ does not satisfy Assumption 4.2. In general, j is given by an integral, and we can overcome this difficulty by considering an approximation φ of ψ obtained by a numerical quadrature.

To conclude, writing $u := u^{\iota+1}$ and $\mathcal{K} := K^{\iota+1}$, problem (R^ι_ν) (or (Q^ι_ν)) could be approximated by a problem (51) which satisfies the conditions of Theorem 4.1 if ψ in (62) could be approximated by a φ, which is lower semicontinuous and convex in the second variable, and satisfies (49) and Assumption 4.2. In this case, the Schwarz methods given by Algorithms 4.1–4.3 are globally convergent with a convergence rate depending on $1/\delta$.

We focus in the following on the application of Algorithms 4.1–4.3 for the solution of the finite element form of problem (Q^ι_ν). Evidently, Theorem 4.1 can be applied for finite element spaces, too. More precisely, we study the convergence of the one- and two-level methods for this problem. As above, these methods are directly obtained from Algorithms 4.1–4.3. We shall see that Assumption 4.1 holds for closed convex sets K_h satisfying a similar property with that given in Property 4.1, in particular for the discretized form of $K^{\iota+1}$. Moreover, we are able to explicitly write the dependence of C_0 on the overlapping and mesh parameters. Also, we give some numerical approximations φ of the functional j for which Assumption 4.2 holds. Therefore, from Theorem 4.1, we can conclude that these methods globally converge for the discretized form of (Q^ι_ν) if conditions (2) and (3) on F, and condition (11) on j hold. Moreover, from the dependence of C_0 on the mesh and domain decomposition parameters, we will conclude that the convergence rate is optimal, i.e. it is similar with that in the case of linear equations, for instance. The convergence rate of the two-level method depends very weakly on the mesh and domain decomposition parameters, and, for some particular choices, it is even independent of them.

4.2.1 One-level method

We consider a simplicial regular mesh partition $\mathcal{T}_h$ (see [8], p. 124, for instance) of mesh size h over the domain $\Omega \subset \mathbb{R}^d$. The domain Ω is decomposed as in (61) with the overlapping parameter δ, and we assume that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$.

We associate to the decomposition (61), some functions $\theta^i \in C^0(\bar{\Omega})$, $\theta^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \cdots, m$, such that

$$\begin{aligned} & 0 \le \theta^i \le 1 \text{ on } \Omega, \\ & \theta^i = 0 \text{ on } \cup_{j=i+1}^m \Omega_j \backslash \Omega_i,\ \theta^i = 1 \text{ on } \Omega_i \backslash \cup_{j=i+1}^m \Omega_j, \end{aligned} \tag{63}$$

and

$$|\partial_{x_k} \theta^i| \le C/\delta, \text{ a.e. in } \Omega, \qquad \forall\, k = 1, \ldots, d \tag{64}$$

Such functions θ^i with the above properties exist (see [18], p. 59, for instance). As in (64), we denote in the following by C a generic constant which does not depend on either the mesh or the domain decomposition.

We consider the piecewise linear finite element space

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\}, \tag{65}$$

and also, for $i = 1, \ldots, m$, we take

$$V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega \backslash \Omega_i\} \tag{66}$$

as some subspaces of V_h corresponding to the domain decomposition $\Omega_1, \ldots, \Omega_m$. The spaces V_h and V_h^i, $i = 1, \ldots, m$, are considered as subspaces of H^1. We denote by $\|\cdot\|_0$ the norm in L^2, and by $\|\cdot\|_1$ and $|\cdot|_1$ the norm and seminorm in H^1, respectively.

We have assumed that the convex set $K^{\iota+1}$ of problem (R_ν^ι) is a particular case of a convex set $\mathcal{K}$ with Property 4.1. We consider that the discretized form of $\mathcal{K} = K^{\iota+1}$ is defined as a subset $K_h \subset V_h$ which satisfies a similar property,

Property 4.2. If $v, w \in K_h$, and if $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$.

Above, we have denoted by L_h the P_1-Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_h$. This property is satisfied, in general by the convex sets which are defined by the values of the functions at the mesh nodes. Like Property 4.1, Property 4.2 is satisfied for convex sets of the one- or two-obstacle type.

Let us consider some continuous functionals $I_\kappa : L^2(\Omega) \to \mathbb{R}$, and assume that the functional φ, the finite element approximation of ψ in (62), is of the form

$$\varphi(u, v) = \sum_{\kappa \in \mathcal{N}_h} I_\kappa(\phi\,(u, v(x_\kappa))) = \sum_{\kappa \in \mathcal{N}_h} \phi_\kappa(u, v) \tag{67}$$

where $\mathcal{N}_h$ is the set of nodes of the mesh partition $\mathcal{T}_h$, $\phi : K_h \times \mathbb{R} \to L^2(\Omega)$ is continuous, and we assume that for any $u \in K_h$

$$I_\kappa(\phi\,(u, \cdot)) : \mathbb{R} \to \mathbb{R}\,, \qquad \kappa \in \mathcal{N}_h\,,$$

are convex functions. For the ease of notation, for all $\kappa \in \mathcal{N}_h$ we have written $\phi_\kappa(u, v) = I_\kappa(\phi(u, v(x_\kappa)))$. We see that ϕ_κ can be viewed as functionals $\phi_\kappa : K_h \times K_h \to \mathbb{R}$ which satisfy

$$\begin{aligned} &\phi_\kappa(u, L_h(\theta v + (1-\theta)w)) \\ \le&\theta(x_\kappa)\phi_\kappa(u, v) + (1 - \theta(x_\kappa))\phi_\kappa(u, w) \end{aligned} \tag{68}$$

for any $u, v, w \in K_h$, and any function $\theta : \bar{\Omega} \to \mathbb{R}$ with the properties $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$.

Now, writing the discretized variant of (R_ν^ι) as a problem of the form (49), we can conclude from the following proposition that the error estimations in Theorem 4.1 hold for the one-level multiplicative Schwarz methods given by Algorithms 4.1-4.3 if F satisfies (2) and (3), and φ satisfies (49). In general, φ satisfies this condition if j satisfies (11).

Proposition 1. *Assumption* 4.1 *holds for the piecewise linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \ldots, m$, and any convex set $K_h \subset V_h$ having Property* 4.2. *Also, Assumption 4.2 on the functional φ of the form (67) is satisfied. The constant in (48) can be written as*

$$C_0 = Cm\left(1 + \frac{m-1}{\delta}\right), \tag{69}$$

where C is independent of the mesh and domain decomposition parameters.

The proof of this proposition can be found in [4] or [5]. It is proved there that Assumptions 4.1 and 4.2 hold if we consider

$$v_i = L_h\left(\theta^i(v - w - \sum_{j=1}^{i-1} v_j) + (1-\theta^i)w_i\right), \quad i = 1, \ldots, m, \tag{70}$$

where the functions θ_i satisfy (63) and (64).

4.2.2 Two-level method

We consider two simplicial regular mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ on the domain $\Omega \subset \mathbb{R}^d$ of mesh sizes h and H, respectively.

As in the previous section, we consider an overlapping decomposition (61), the mesh partition $\mathcal{T}_h$ of Ω supplying a mesh partition for each Ω_i, $1 \le i \le m$. Also, we assume that the overlapping size is δ. In addition, we suppose that there exists a constant C, independent of both meshes, such that the diameter of the connected components of each Ω_i are less than CH. The domain Ω may be different from

$$\Omega_0 = \bigcup_{\tau \in \mathcal{T}_H} \tau, \tag{71}$$

but we assume that the mesh $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$ on Ω_0, and if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$ then it also lies on $\partial\Omega$. Finally, we suppose that there exists a constant C, independent of both meshes, such that

$$\mathrm{dist}(x, \Omega_0) \le CH \tag{72}$$

for any node x of $\mathcal{T}_h$.

Now, besides the spaces V_h and V_h^i, $i = 1, \ldots, m$, defined in (65) and (66), we introduce the continuous, piecewise linear finite element space corresponding to the H-level,

$$V_H^0 = \left\{ v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0 \right\}, \qquad (73)$$

where the functions v are extended with zero in $\Omega \backslash \Omega_0$. The convex set $K_h \subset V_h$ is defined as a subset of V_h having Property 4.2.

The two-level Schwarz methods are also obtained from Algorithms 4.1–4.3 in which we take $V = V_h$, $\mathcal{K} = K_h$, and the subspaces $V_0 = V_H^0$, $V_1 = V_h^1$, $V_2 = V_h^2, \ldots, V_m = V_h^m$. As in the previous section, the spaces V_h, V_H^0, V_h^1, $V_h^2, \ldots, V_h^m$, are considered as subspaces of H^1. We note that, this time, the decomposition of the domain Ω contains m overlapping subdomains, but we use $m+1$ subspaces of V, $V_0, V_1, \ldots, V_m$, in Algorithms 4.1–4.3. Assuming that F satisfies (2) and (3) and φ satisfies (49), if we prove that Assumption 4.1, written for $m+1$ subspaces, is satisfied for the previous choice of the convex set $\mathcal{K}$ and the subspaces $V_0, V_1, \ldots, V_m$ of V, and the functional φ of the form (67) satisfies Assumption 4.2, we can conclude that these Algorithms 4.1–4.3 converge. To this end, we consider the operator $I_H : V_h \to V_H^0$, which has been introduced in [2] and has the following properties (see Lemma 4.2 in [2]) for any $v \in V_h$:

$$\|I_H v - v\|_0 \le C H C_d(H,h) |v|_1 \qquad (74)$$

and

$$\|I_H v\|_0 \le C \|v\|_0 \text{ and } |I_H v|_1 \le C C_d(H,h) |v|_1, \qquad (75)$$

where

$$C_d(H,h) = \begin{cases} 1 & \text{if } d = 1 \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{1}{2}} & \text{if } d = 2 \\ \left(\frac{H}{h}\right)^{\frac{1}{2}} & \text{if } d = 3. \end{cases} \qquad (76)$$

Moreover, for any $x \in \Omega$, we have

$0 \le I_H v(x) \le v(x)$ if $v(x) > 0$,
$0 \ge I_H v(x) \ge v(x)$ if $v(x) < 0$,
and $I_H v = 0$ on $\tau \in \mathcal{T}_H$ if there exists a $x \in \tau$ such that $v(x) = 0$

for any $v \in V_h$. Consequently, writing

$$\theta_v(x) = \begin{cases} \frac{I_H v(x)}{v(x)} & \text{if } v(x) \ne 0 \\ 0 & \text{if } v(x) = 0, \end{cases} \qquad (77)$$

then $\theta_v \in C^0(\bar{\Omega})$, $\theta_v|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, $0 \leq \theta_v \leq 1$, and

$$I_H v = \theta_v v$$

for any $v \in V_h$.

Now, we can prove the following proposition which shows that the constant C_0 in Assumption 4.1 is independent of the mesh and domain decomposition parameters if H/δ and H/h are constant.

Proposition 2. *Assumption* 4.1 *is satisfied for the piecewise linear finite element spaces $V = V_h$ and $V_0 = V_H^0$, $V_i = V_h^i$, $i = 1, \ldots, m$, defined in (65), (66), and (73), respectively, and any convex set $\mathcal{K} = K_h$ with Property 4.2. Also, Assumption 4.2 for the functional φ of the form (67) is satisfied. The constant in (48) of Assumption 4.1 can be taken of the form*

$$C_0 = Cm\left(1 + (m-1)\frac{H}{\delta}\right) C_d(H, h), \tag{78}$$

where C is independent of the mesh and domain decomposition parameters, and $C_d(H, h)$ is given in (76).

A detailed proof of this proposition can be found in [4] or [5]. By means of I_H and the functions θ^i, $i = 1, \ldots, m$, with properties (63) and (64), we define

$$v_0 = w_0 + I_H(v - w - w_0). \tag{79}$$

and

$$v_i = L_h(\theta^i(v - w - v_0 - \sum_{j=1}^{i-1} v_j) + (1 - \theta^i)w_i), \tag{80}$$

for $i = 1, \ldots, m$. Using properties (74) and (75) of the operator I_H, it is proved that $v_0, v_1, \ldots, v_m$, defined in (79) and (80), satisfy Assumption 4.1 with the constant C_0 given in (78). Also, using (68) in which θ is replaced by θ^i or by functions of the type θ_v, defined in (77), it is proved that Assumption 4.2 hold, too.

Remark 4. In this section, we have assumed that the functional φ is of the form (67). With similar proofs, we can show that Propositions 1 and 2 also hold if we replace the functional $\varphi(u, v)$ of the form (67) with

$$\varphi(u, v) = \sum_{\kappa \in \mathcal{N}_h} I_\kappa(\phi(u(x_\kappa), v(x_\kappa))) = \sum_{k \in \mathcal{N}_h} \phi_\kappa(u, v), \tag{81}$$

where $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $I_\kappa : \mathbb{R} \to \mathbb{R}$, $\kappa \in \mathcal{N}_h$, are continuous and $I_\kappa(\phi(r, \cdot)) : \mathbb{R} \to \mathbb{R}$, $\kappa \in \mathcal{N}_h$, are convex functions for any $r \in \mathbb{R}$. For the writing simplicity, we have written $\phi_\kappa(u,v) = I(\phi(u(x_\kappa), v(x_\kappa)))$, $\kappa \in \mathcal{N}_h$. In general, (67) or (81) represent numerical approximations of some integrals. Concerning to condition (49) imposed on φ of the form (67) or (81), we have to check it for each particular problem we solve.

The results of this subsection have referred to problems in H^1 with Dirichlet boundary conditions. We point out that similar results can be obtained for problems in $(H^1)^d$ or problems with mixed boundary conditions.

Acknowledgments

The support of this work by "Laboratoire Européen Associé CNRS Franco-Roumain de Mathématiques et Modélisation" LEA Math-Mode is gratefully acknowledged.

References

[1] L. Badea, *Convergence rate of a multiplicative Schwarz method for strongly nonlinear inequalities*, in "Analysis and optimization of differential systems" (eds. V.Barbu, I. Lasiecka, D. Tiba and C. Varsan), Kluwer Academic Publishers, (2003), 31–42 (also available from http://imar.ro/~lbadea/pub.html).

[2] L. Badea, *Convergence rate of a Schwarz multilevel method for the constrained minimization of non-quadratic functionals*, SIAM J. Numer. Anal., **44** (2006), 449–477.

[3] L. Badea, *Schwarz methods for inequalities with contraction operators*, J. Comp. Appl. Math., **215** (2008), 196–219 (doi:10.1016/j.cam.2007.04.004).

[4] L. Badea and R. Krause, *One- and two-level Schwarz methods for variational inequalities of the second kind and their application to frictional contact*, Numer. Math., DOI: 10.1007/s00211-011-0423-y, online published on 5 November 2011.

[5] L. Badea and R. Krause, *One- and two-level multiplicative Schwarz methods for variational and quasi-variational inequalities of the second kind: Part I - general convergence results*, INS Preprint, no. 0804, Institute for Numerical Simulation, University of Bonn, June 2008.

[6] A. Capatina and M. Cocou, *Internal approximation of quasi-variational inequalities*, Numer. Math., **59** (1991), 385–398.

[7] A. Capatina, M. Cocou and M. Raous, *A class of implicit variational inequalities and applications to frictional contact*, Math. Meth. Appl. Sci., **32** (2009), 1804–1827.

[8] P. G. Ciarlet, "The Finite Element Method for Elliptic Problems," North-Holland, Amsterdam, 1978.

[9] M. Cocou, E. Pratt and M. Raous, *Formulation and approximation of quasistatic frictional contact*, Int. J. Engrg. Sci., **34** (1996), 783–798.

[10] R. Glowinski, "Numerical Methods for Nonlinear Variational Problems," Springer-Verlag, New York, 1984.

[11] R. Glowinski, J. L. Lions and R. Trémolières, "Analyse numérique des inéquations variationnelles," Dunod, Paris, 1976.

[12] R. Kornhuber, *Monotone multigrid methods for elliptic variational inequalities I*, Numer. Math., **69** (1994), 167–184.

[13] R. Kornhuber, *Monotone multigrid methods for elliptic variational inequalities II*, Numer. Math., **72** (1996), 481–499.

[14] R. Kornhuber, "Adaptive Monotone Multigrid Methods for Nonlinear Variational Problems," Teubner-Verlag, 1997.

[15] J. Mandel, *A multilevel iterative method for symmetric, positive definite linear complementarity problems*, Appl. Math. Opt., **11** (1984), 77–95.

[16] J. Mandel, *Etude algébrique d'une méthode multigrille pour quelques problèmes de frontière libre*, C. R. Acad. Sci. Série I, **298** (1984), 469–472.

[17] M. Raous, L. Cangémi and M. Cocou, *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Meth. Appl. Mech. Engrg., **177** (1999), 383–399.

[18] A. Toselli and O. Widlund, "Domains Decomposition Methods - Algorithms and Theory," Springer-Verlag, Berlin, 2005.